CLASS IX (Mathematics)

Chapter - 3 Expansions

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Variable : Letters of English alphabet e.g. a, b, c, x, y, z, ...

Constant : A number with a definite value e.g. 1, 6, 9, 25, 82...

Algebraic Expression : Combination of constants and variables Each term is separated by '+' or '−' signs.

Expansions (Special Products) :-

Multiplication of any algebraic expression by itself or any other algebraic expression is called its expansion.

Expansion of $(a+b)^2$ and $(a-b)^2$

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2 \quad \text{or} \quad \boxed{a^2+b^2+2ab}$$

$$(a-b)^2 = (a-b)(a-b) = a(a-b) - b(a-b)$$
$$= a^2 - ab - ab + b^2$$
$$= a^2 - 2ab + b^2 \quad \text{or} \quad \boxed{a^2+b^2-2ab}$$

Example1: Find the squares of the following :-

1. $(2x+y)^2 = (2x)^2 + (y)^2 + 2 \times 2x \times y$
$= 4x^2 + y^2 + 4xy$

2. $\left(\frac{2}{7}x + \frac{3}{5}y\right)^2 = \left(\frac{2}{7}x\right)^2 + \left(\frac{3}{5}y\right)^2 + 2 \times \frac{2}{7}x \times \frac{3}{5}y$
$= \frac{4}{49}x^2 + \frac{9}{25}y^2 + \frac{12}{35}xy$

3. $(2a-3b)^2 = (2a)^2 + (3b)^2 - 2 \times 2a \times 3b$
$= 4a^2 + 9b^2 - 12ab$

4. $\left(\frac{a}{6} - \frac{3}{a}\right)^2 = \left(\frac{a}{6}\right)^2 + \left(\frac{3}{a}\right)^2 - 2 \times \frac{\cancel{a}}{\cancel{6}_2} \times \frac{\cancel{3}}{\cancel{a}}$
$= \frac{a^2}{36} + \frac{9}{a^2} - 1$

5. $\left(x+\frac{1}{2x}\right)^2 = (x)^2 + \left(\frac{1}{2x}\right)^2 + 2 \times \cancel{x} \times \frac{1}{\cancel{2x}}$

$= x^2 + \frac{1}{4x^2} + 1$

6. $\left(3x-\frac{1}{x}\right)^2 = (3x)^2 + \left(\frac{1}{x}\right)^2 - 2 \times 3\cancel{x} \times \frac{1}{\cancel{x}}$

$= 9x^2 + \frac{1}{x^2} - 6$

Expansions of $(a+b)^3$ and $(a-b)^3$

$(a+b)^3 = (a+b)(a+b)(a+b) = (a+b)(a+b)^2$

$= (a+b)(a^2+b^2+2ab)$

$= a(a^2+b^2+2ab) + b(a^2+b^2+2ab)$

$= a^3 + ab^2 + 2a^2b + a^2b + b^3 + 2ab^2$

$= a^3 + b^3 + 3a^2b + 3ab^2$

or $\boxed{a^3+b^3+3ab(a+b)}$

$(a-b)^3 = (a-b)(a-b)(a-b) = (a-b)(a-b)^2$

$= (a-b)(a^2+b^2-2ab)$

$= a(a^2+b^2-2ab) - b(a^2+b^2-2ab)$

$= a^3 + ab^2 - 2a^2b - a^2b - b^3 + 2ab^2$

$= a^3 - b^3 - 3a^2b + 3ab^2$

or $\boxed{a^3-b^3-3ab(a-b)}$

Example 2: Find the cubes of the following:-

1. $(2a+3b)^3 = (2a)^3 + (3b)^3 + 3 \times 2a \times 3b\ (2a+3b)$

$= 8a^3 + 27b^3 + 18ab(2a+3b)$

$= 8a^3 + 27b^3 + 36a^2b + 54ab^2$

2. $(x-4y)^3 = x^3 - (4y)^3 - 3 \times x \times 4y\ (x-4y)$

$= x^3 - 64y^3 - 12xy(x-4y)$

$= x^3 - 64y^3 - 12x^2y + 48xy^2$

3. $\left(4p - \frac{1}{3q}\right)^3 = (4p)^3 - \left(\frac{1}{3q}\right)^3 - 3 \times 4p \times \frac{1}{3q}\left(4p - \frac{1}{3q}\right)$

$= 64p^3 - \frac{1}{27q^3} - \frac{4p}{q}\left(4p - \frac{1}{3q}\right)$

$= 64p^3 - \frac{1}{27q^3} - \frac{16p^2}{q} + \frac{4p}{3q^2}$

4. $\left(\frac{3}{x} - \frac{1}{2x}\right)^3 = \left(\frac{3}{x}\right)^3 - \left(\frac{1}{2x}\right)^3 - 3 \times \frac{3}{x} \times \frac{1}{2x}\left(\frac{3}{x} - \frac{1}{2x}\right)$

$= \frac{27}{x^3} - \frac{1}{8x^3} - \frac{9}{2x^2}\left(\frac{3}{x} - \frac{1}{2x}\right)$

$= \frac{27}{x^3} - \frac{1}{8x^3} - \frac{27}{2x^3} + \frac{9}{4x^3}$

5. $(4a^2 - 3b)^3 = (4a^2)^3 - (3b)^3 - 3 \times 4a^2 \times 3b(4a^2 - 3b)$

$= 64a^6 - 27b^3 - 36a^2b(4a^2 - 3b)$

$= 64a^6 - 27b^3 - 144a^4b + 108a^2b^2$

Expansion of $(a+b+c)^2$

$(a+b+c)^2 = (a+b+c)(a+b+c)$

$= a(a+b+c) + b(a+b+c) + c(a+b+c)$

$= a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

or $\boxed{a^2 + b^2 + c^2 + 2(ab + bc + ac)}$

Similarly, $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$

$(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$

$(a-b-c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ac$

Example 3: Expand the following :-

1. $(x + 2y + z)^2 = (x)^2 + (2y)^2 + (z)^2 + 2 \times x \times 2y + 2 \times 2y \times z + 2 \times z \times x$

$= x^2 + 4y^2 + z^2 + 4xy + 4yz + 2xz$

2. $(2x+3y-4z)^2 = (2x)^2+(3y)^2+(4z)^2+2\times 2x\times 3y$
$+2\times 3y\times 4z - 2\times 2x\times 4z$
$= 4x^2+9y^2+16z^2+12xy-24yz-16xz$

3. $\left(\frac{2}{3}x-\frac{3}{2x}-1\right)^2 = \left(\frac{2}{3}x\right)^2+\left(\frac{3}{2x}\right)^2+(1)^2-2\times\frac{2}{3}x\times\frac{3}{2x}$
$+2\times\frac{3}{2x}\times 1 - 2\times\frac{2}{3}x\times 1$
$= \frac{4x^2}{9}+\frac{9}{4x^2}+1-2+\frac{3}{x}-\frac{4x}{3}$
$= \frac{4}{9}x^2+\frac{9}{4x^2}+\frac{3}{x}-\frac{4x}{3}-1$

4. $\left(3x-\frac{1}{x}+5\right)^2 = (3x)^2+\left(\frac{1}{x}\right)^2+(5)^2-2\times 3x\times\frac{1}{x}$
$-2\times\frac{1}{x}\times 5+2\times 5\times 3x$
$= 9x^2+\frac{1}{x^2}+25-6-\frac{10}{x}+30x$
$= 9x^2+\frac{1}{x^2}-\frac{10}{x}+30x+19$

Expansion of $(a+b)(a-b)$

$(a+b)(a-b) = a(a-b)+b(a-b)$
$= a^2-\cancel{ab}+\cancel{ab}-b^2$
$= \boxed{a^2-b^2}$

Example 4: Simplify the following :-

1. $(5x-9)(5x+9) = (5x)^2-(9)^2 = 25x^2-81$
2. $\left(x+\frac{7}{2}y\right)\left(x-\frac{7}{2}y\right) = (x)^2-\left(\frac{7}{2}y\right)^2 = x^2-\frac{49}{4}y^2$
3. $\left(3a-\frac{5}{2}b\right)\left(3a+\frac{5}{2}b\right) = (3a)^2-\left(\frac{5}{2}b\right)^2 = 9a^2-\frac{25}{4}b^2$
4. $(x+y)(x-y)(x^2+y^2) = (x^2-y^2)(x^2+y^2)$
$= (x^2)^2-(y^2)^2$
$= x^4-y^4$

Example 5: Using standard formulae, evaluate :-

(i) $(107)^2 = (100+7)^2$ [Using formula $(a+b)^2 = a^2+b^2+2ab$]

$= (100)^2 + (7)^2 + 2\times100\times7$

$= 10000 + 49 + 1400 = 11449$ (Ans)

(ii) $(20.6)^2 = (20+0.6)^2$

$= (20)^2 + (0.6)^2 + 2\times20\times0.6$

$= 400 + 0.36 + 24 = 424.36$

(iii) $(996)^2 = (1000-4)^2$ [Using formula $(a-b)^2 = a^2+b^2-2ab$]

$= (1000)^2 + (4)^2 - 2\times1000\times4$

$= 1000000 + 16 - 8000 = 992016$

(iv) $(9.7)^2 = (10-0.3)^2$

$= (10)^2 + (0.3)^2 - 2\times10\times0.3$

$= 100 + 0.09 - 6 = 94.09$

(v) $124\times116 = (120+4)(120-4)$ [Using formula $(a+b)(a-b) = a^2-b^2$]

$= (120)^2 - (4)^2$

$= 14400 - 16 = 14384$

(vi) $3.99\times4.01 = (4-0.01)(4+0.01)$

$= (4)^2 - (0.01)^2$

$= 16 - 0.0001 = 15.9999$

Standard formulae :-

1. $(a+b)^2 = a^2+b^2+2ab$
2. $(a-b)^2 = a^2+b^2-2ab$
3. $(a+b)^3 = a^3+b^3+3ab(a+b)$
4. $(a-b)^3 = a^3-b^3-3ab(a-b)$
5. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$
6. $(a+b)(a-b) = a^2-b^2$

Examples based on application of formulae

Example 6: If $a-b=8$ and $ab=5$, find a^2+b^2

Solution: We know that, $(a-b)^2 = a^2+b^2-2ab$

$\Rightarrow (8)^2 = a^2+b^2-2\times5$

$\Rightarrow 64 = a^2+b^2-10 \quad \Rightarrow a^2+b^2 = 54$

Example 7: If $a^2+b^2+c^2=5$ and $ab+bc+ca=10$, find the value of $a+b+c$

Solution: We know that, $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$\Rightarrow (a+b+c)^2 = 5+2\times10 = 25$

$\therefore (a+b+c) = \sqrt{25} = \pm5$

Example 8: If $x+y=6$ and $xy=8$, find x^3+y^3

Solution: We know that, $(a+b)^3 = a^3+b^3+3ab(a+b)$

$\Rightarrow (x+y)^3 = x^3+y^3+3xy(x+y)$

$\Rightarrow (6)^3 = x^3+y^3+3\times8\times6$

$\Rightarrow 216 = x^3+y^3+144$

$\Rightarrow x^3+y^3 = 72$

Example 9: If $a^2+\frac{1}{a^2}=7$, find the values of

(i) $\left(a+\frac{1}{a}\right)$ (ii) $\left(a-\frac{1}{a}\right)$ (iii) $\left(a^2-\frac{1}{a^2}\right)$

Solution: (i) $\left(a+\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}+2\times\cancel{a}\times\frac{1}{\cancel{a}}$

$= 7+2$

$= 9$

$\therefore a+\frac{1}{a} = \sqrt{9} = \pm3$

(ii) $\left(a-\frac{1}{a}\right)^2 = a^2+\frac{1}{a^2}-2\times\cancel{a}\times\frac{1}{\cancel{a}}$

$= 7-2$

$= 5$

$\therefore \left(a-\frac{1}{a}\right) = \pm\sqrt{5}$

(iii) $a^2-\frac{1}{a^2} = \left(a+\frac{1}{a}\right)\left(a-\frac{1}{a}\right) = \pm(3\times\sqrt{5}) = \pm3\sqrt{5}$

Example 10: If $x - \frac{1}{x} = 5$, find the value of

(i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$

Solution: (i) We know that, $(a-b)^2 = a^2 + b^2 - 2ab$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 \times \cancel{x} \times \frac{1}{\cancel{x}}$$

$$\Rightarrow (5)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 25 + 2$$

$$\therefore x^2 + \frac{1}{x^2} = 27$$

(ii) $\left(x^2 + \frac{1}{x^2}\right)^2 = (27)^2$ [Squaring both sides]

$$(x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 \times \cancel{x^2} \times \frac{1}{\cancel{x^2}} = 729$$

$$x^4 + \frac{1}{x^4} = 729 - 2$$

$$\therefore x^4 + \frac{1}{x^4} = 727$$

Example 11: If $\frac{x^2+1}{x} = 2\frac{1}{2}$, find the values of

(i) $x - \frac{1}{x}$ (ii) $x^3 - \frac{1}{x^3}$

Solution: (i) $\frac{x^2+1}{x} = 2\frac{1}{2} \Rightarrow \frac{x^2}{x} + \frac{1}{x} = \frac{5}{2}$

$\Rightarrow x + \frac{1}{x} = \frac{5}{2}$ on squaring both sides, we get

$$\left(x + \frac{1}{x}\right)^2 = \left(\frac{5}{2}\right)^2 \Rightarrow x^2 + \frac{1}{x^2} + 2 \times \cancel{x} \times \frac{1}{\cancel{x}} = \frac{25}{4}$$

$$\Rightarrow x^2 + \frac{1}{x^2} = \frac{25}{4} - 2 = \frac{25-8}{4} = \frac{17}{4}$$

$$\therefore x^2 + \frac{1}{x^2} = \frac{17}{4} \text{ --------- (i)}$$

Now, $\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 \times \cancel{x} \times \frac{1}{\cancel{x}}$

$$\left(x - \frac{1}{x}\right)^2 = \frac{17}{4} - 2 = \frac{9}{4}$$ Using (i)

$$\therefore x - \frac{1}{x} = \sqrt{\frac{9}{4}} = \pm \frac{3}{2}$$

(ii) $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3 \times x \times \frac{1}{x}\left(x - \frac{1}{x}\right)$

$= x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$

Case I When $x - \frac{1}{x} = \frac{3}{2}$

Then, $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x-\frac{1}{x}\right)$

$\Rightarrow \left(\frac{3}{2}\right)^3 = x^3 - \frac{1}{x^3} - 3 \times \frac{3}{2}$

$\Rightarrow \frac{27}{8} + \frac{9}{2} = x^3 - \frac{1}{x^3}$

$\Rightarrow x^3 - \frac{1}{x^3} = \frac{27+36}{8} = \frac{63}{8} = 7\frac{7}{8}$

Case II When $x - \frac{1}{x} = -\frac{3}{2}$

Then, $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x-\frac{1}{x}\right)$

$\Rightarrow \left(-\frac{3}{2}\right)^3 = x^3 - \frac{1}{x^3} - 3 \times \left(-\frac{3}{2}\right)$

$\Rightarrow -\frac{27}{8} - \frac{9}{2} = x^3 - \frac{1}{x^3}$

$\Rightarrow x^3 - \frac{1}{x^3} = \frac{-27-36}{8} = -\frac{63}{8} = -7\frac{7}{8}$

Hence, $x^3 - \frac{1}{x^3} = \pm 7\frac{7}{8}$

An Important Result

If $a+b+c=0$, then $a^3+b^3+c^3 = 3abc$

Proof: Given, $a+b+c = 0$

$\Rightarrow a+b = -c$ ---------- (i)

$(a+b)^3 = (-c)^3$ [cubing both sides]

$\Rightarrow a^3+b^3+3ab(a+b) = -c^3$

$\Rightarrow a^3+b^3+3ab \times (-c) = -c^3$ [using (i)

$\Rightarrow a^3+b^3-3abc = -c^3$

$\Rightarrow a^3+b^3+c^3 = 3abc$

Hence proved.

— (8) —