Tender Heart High School,Sector 33B,Chd.

Class 10, Mathematics

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Chapter-8 Remainder Theorem & Factor Theorem

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Introduction:-

In this section we will study particular type of algebraic expression, called polynomial.

We will learn (i) types of polynomials
(ii) remainder theorem (iii) factor theorem
(iv) factorisation of polynomials

POLYNOMIAL IN ONE VARIABLE

An algebraic expression in which exponents on the variable(s) are whole numbers only. are called polynomial.

Each term of a polynomial must be in the form of

ax^n where, 'a' is coefficient
'n' is exponent / power

Examples:-

(i) $x^3 + 3x^{3/2} + 5x - 2$ is not a polynomial because the exponent of second term is $3/2$ which is not a whole number.

(ii) $x^2 + 2\sqrt{x} + 3$ is not a polynomial as exponent on second term is $1/2$, which is not a whole number.

(iii) $2x^3 + 3x^2 + 5$ is a polynomial as each term follows the condition of term of polynomial.

General form of a polynomial in one variable

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

OR $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots \cdots + a_n x^n$

OR $p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_{n-1} x + a_n$

where, $a_0, a_1, a_2, \cdots a_n$ are real numbers and n is a non-negative integer, is called a polynomial in the variable x.

NOTE:- Polynomial in one variable is denoted by $p(x)$, $q(x)$, $g(x)$, $h(x)$, $f(x)$, etc.

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Types of Polynomial	Degree	Name of Polynomial
$p(x) = c$	0	Constant
$p(x) = ax+b$, $a \neq 0$	1	Linear
$p(x) = ax^2+bx+c$, $a \neq 0$	2	Quadratic
$p(x) = ax^3+bx^2+cx+d$, $a \neq 0$	3	Cubic
$p(x) = ax^4+bx^3+cx^2+dx+e$, $a \neq 0$	4	bi-quadratic

REMAINDER THEOREM

In this section we will discuss about remainder left on dividing a polynomial of degree one or more by a linear polynomial. To understand this in more better way first we will look at procedure of division of two positive integers.

Let us consider to divide 17 by 5

$$\begin{array}{r|l} & 3 \rightarrow \text{Quotient} \\ \text{Divisor} \leftarrow 5 & 17 \rightarrow \text{Dividend} \\ & -15 \\ \hline & 2 \rightarrow \text{Remainder} \end{array} \qquad \begin{array}{r|l} & q(x) \\ g(x) & p(x) \\ & \vdots \\ \hline & r(x) \end{array}$$

Now, $17 = 5 \times 3 + 2$

Dividend = Divisor × Quotient + Remainder

$$p(x) = g(x)\, q(x) + r(x)$$

Here divisior is a linear polynomial, so we can take $g(x) = x - a$

i.e. when $p(x)$ will be divided by $x-a$ then degree of $r(x)$ will be less tha that of $x-a$

$$\Rightarrow \quad p(x) = (x-a)\, q(x) + r$$

[Since the degree of $(x-a)$ is 1, the remainder is a constant]

On putting $x = a$, we get

$$p(a) = (a-a)\, q(a) + r$$

$$p(a) = r \text{ (remainder)}$$

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Important points:-

To find remainder equate the divisor to zero which gives value of x.

1) If a polynomial $p(x)$ is divided by $(x+a)$, the remainder is the value of $p(x)$ at $x = -a$ i.e. $p(-a)$

2) If a polynomial $p(x)$ is divided by $(ax-b)$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right)$

Similarly, if $ax+b$ then $x = \frac{-b}{a}$ i.e. $p\left(\frac{-b}{a}\right)$

Example 1:-

Find the remainder when $p(x) = x^3+x^2+2x+3$ is divided by $x+2$

Solution:- When $p(x)$ is divided by $(x+2)$, then remainder is $p(-2)$

$\Rightarrow p(-2) = (-2)^3+(-2)^2+2(-2)+3$

$= -8+4-4+3 = -5$

Therefore, remainder is -5

Example 2:-

Find the remainder when $p(x) = x^3+7x+3$ is divided by $2x-1$

Solution:- When $p(x)$ is divided by $(2x-1)$, then remainder is $p\left(\frac{1}{2}\right)$ [By remainder theorem]

$\Rightarrow p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3+7\left(\frac{1}{2}\right)+3$

$= \frac{1}{8}+\frac{7}{2}+3 = \frac{53}{8}$

Therefore, remainder is $\frac{53}{8}$

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Example 3:-
If the polynomial $3x^2-2x+k$ is divided by $x+3$, the remainder is 5. Find the value of 'k'.
Solution:- Let $p(x) = 3x^2-2x+k$
Put $x+3=0 \Rightarrow x=-3$
Now by remainder theorem $p(-3)$ is the remainder
$\Rightarrow p(-3) = 3(-3)^2-2\times(-3)+k$
$\Rightarrow 5 = 3\times 9+6+k$
$\Rightarrow 5 = 33+k \Rightarrow k = -28$

Example 4:- If the polynomials ax^3+4x+5 and x^3-4x^2+a leave the same remainder when divided by $x-2$, find the value of a.
Solution:- By remainder theorem put $x-2=0 \Rightarrow x=2$
So, $p(2) = a(2)^3+4(2)+5 = 8a+13$ ---(i)
and remainder left on dividing x^3-4x^2+a by $x-2$ is. $p(2) = 2^3-4(2)^2+a = 8-16+a$
$= a-8$ ---(ii)
Since both remainders are equal, so from (i) and (ii), we get $8a+13 = a-8$
$\Rightarrow 7a = -21 \Rightarrow a = \frac{-21}{7} = -3$

Therefore, $a = -3$

Example 5:- Given that px^2+qx+6 leaves the remainder 1 on division by $2x+1$ and $2qx^2+6x+p$ leaves the remainder 2 on division by $3x-1$. Find p and q.
Solution:- Put $2x+1=0 \Rightarrow 2x=-1 \Rightarrow x = \frac{-1}{2}$
Let $f(x) = px^2+qx+6$
Then, $f\left(\frac{-1}{2}\right) = p\left(\frac{-1}{2}\right)^2+q\left(\frac{-1}{2}\right)+6$

contd...

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$\Rightarrow 1 = p \times \frac{1}{4} - \frac{q}{2} + 6 \quad \Rightarrow 1-6 = \frac{p-2q}{4}$

$\Rightarrow -5 \times 4 = p - 2q \quad \Rightarrow p - 2q = -20$ ----(i)

Also, Let $g(x) = 2qx^2 + 6x + p$

then, $g\left(\frac{1}{3}\right) = 2q\left(\frac{1}{3}\right)^2 + 6\left(\frac{1}{3}\right) + p$

$\Rightarrow \quad 2 = \frac{2q}{9} + 2 + p \quad \Rightarrow p = \frac{-2q}{9}$ ----(ii)

Put value of $p = \frac{-2q}{9}$ in equation (i), we get

$\frac{-2q}{9} - 2q = -20 \Rightarrow \frac{2q+18q}{9} = 20 \Rightarrow \frac{20q}{9} = 20$

$\Rightarrow q = 9$ Now, from (ii), $p = \frac{-2 \times 9}{9} = -2$

Therefore, $q = 9$ and $p = -2$

Example 6:- What number should be added to $2x^3 - 3x^2 - 8x$ so that the resulting polynomial leaves the remainder 10 when divided by $2x+1$?

Solution:- Let the number to be added be 'k'

and $f(x) = 2x^3 - 3x^2 - 8x + k$

By remainder theorem, when $f(x)$ is divided by $2x+1$ then remainder is $f\left(-\frac{1}{2}\right)$

$\Rightarrow f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) + k$

$10 = -\frac{2}{8} - \frac{3}{4} + \frac{8}{2} + k$

$10 = -\frac{1}{4} - \frac{3}{4} + 4 + k \Rightarrow 3 + k = 10$

$\Rightarrow k = 7$

Hence, the number to be added is 7.

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FACTOR THEOREM

If $p(x)$ is a polynomial and 'a' is a real number, then $(x-a)$ is a factor of $p(x)$ iff $p(a)=0$
i.e. when $p(x)$ is divided by $(x-a)$, then $p(a)=0$

Example 7:- Determine which of the following polynomials has $(x+1)$ a factor :-

(i) x^3+x^2+x+1

Put $x+1=0 \Rightarrow x=-1$

Let $p(x) = x^3+x^2+x+1$

$p(-1) = (-1)^3+(-1)^2+(-1)+1 = -1+1+0=0$

Since, $p(-1)=0$ so, by factor theorem $(x+1)$ is a factor of $p(x)$

(ii) $x^4+3x^3+3x^2+x+1$

Let $p(x) = x^4+3x^3+3x^2+x+1$

$$p(-1) = (-1)^4+3(-1)^3+3(-1)^2+(-1)+1$$
$$= 1+3(-1)+3\times1+0$$
$$= 1-3+3 = 1$$

Since, $p(-1)\neq 0$ so, by factor theorem $(x+1)$ is not a factor of $p(x)$

Example 8:- Show that $(x+3)$ is a factor of $2x^2-x-21$. Hence, factorise $2x^2-x-21$.

Solution:- Let $p(x) = 2x^2-x-21$

Put $x+3=0 \Rightarrow x=-3$

$$p(-3) = 2(-3)^2-(-3)-21$$
$$= 18+3-21$$
$$= 0$$

$$\begin{array}{r|l} & 2x-7 \\ x+3 & 2x^2-x-21 \\ & \underline{\overset{-}{2x^2}+\overset{-}{6x}} \\ & -7x-21 \\ & \underline{\overset{+}{-7x}\ \overset{+}{-21}} \\ & \times \end{array}$$

So, $(x+3)$ is a factor of $p(x)$

Dividing $2x^2-x-21$ by $(x+3)$, we get $(2x-7)$ as quotient and remainder $=0$

Therefore, $2x^2-x-21 = (x+3)(2x-7)$

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Example 9 Use factor theorem to factorise completely
$$p(x) = x^3 + x^2 - 4x - 4$$

Solution:- Here, we have to first find one value of x, for which $f(x) = 0$ by hit and trial method. For this we see the constant term 4 and consider its factors $\pm 1, \pm 2, \pm 4$

Putting these factors as the values of x, one by one in $p(x)$, we find the value of x for which $p(x) = 0$. That gives us the factor of $p(x)$. The rest we find by long division method.

Now, $p(x) = x^3 + x^2 - 4x - 4$

$p(-1) = (-1)^3 + (-1)^2 - 4(-1) - 4$

$= -1 + 1 + 4 - 4 = 0 \quad \therefore p(-1) = 0$

Also, $p(2) = (2)^3 + (2)^2 - 4(2) - 4$

$= 8 + 4 - 8 - 4 = 0 \quad \therefore p(2) = 0$

Here, we consider only one factor

Let us take $p(-2) = 0 \Rightarrow x - 2$ is a factor of $p(x)$

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x-2 & x^3 + x^2 - 4x - 4 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & 3x^2 - 4x \\ & 3x^2 - 6x \\ & - \quad + \\ \hline & 2x - 4 \\ & 2x - 4 \\ & - \quad + \\ \hline & \times \end{array}$$

On dividing $x^3 + x^2 - 4x - 4$ by $x - 2$, we get $x^2 + 3x + 2$ as the quotient and remainder $= 0$

Now, $x^2 + 3x + 2 = x^2 + x + 2x + 2$

$= x(x+1) + 2(x+1)$

$= (x+1)(x+2)$

Hence, $x^3 + x^2 - 4x - 4 = (x-2)(x+1)(x+2)$